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The initial-value problem for the dKP hierarchy

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Abstract
The solution corresponding to each initial condition for the dispersionless KP hierarchy can be found from the integration of a Hamilton–Jacobi equation by means of a transformation of coordinates. The solution is explicitly determined in a parametric form as well as a power series in a deformation parameter. In addition, a twistor formulation of the solution is given.

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1. Introduction

Let $\mathfrak{g}$ be the Lie algebra of holomorphic functions $f(p, x)$ on the domain $E = D \times \mathbf{C}$, for an annulus $D = \{p \in \mathbf{C} : r_- < |p| < r_+\}$. In this Lie algebra the commutator is given by the Poisson bracket $\{f, g\} = f_p g_x - f_x g_p$, where f_p, f_x denote partial derivatives of the function $f(p, x)$. According to Laurent's theorem, $f(p, x)$ admits the representation

$$f(p,x) = \frac{1}{2\pi \mathrm{i}} \int_{C_+} \frac{f(z,x)}{z-p}\,\mathrm{d}z - \frac{1}{2\pi \mathrm{i}} \int_{C_-} \frac{f(z,x)}{z-p}\,\mathrm{d}z,$$

which gives rise to a decomposition of $\mathfrak{g}$ as the direct sum of subalgebras $\mathfrak{g}_\pm$, $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. These are the algebras of functions in $\mathfrak{g}$ that extend analytically, in the variable p, to zero and infinity respectively. One has $f = f_+ - f_-$, where f_+ and f_- are given by the integrals in Laurent's decomposition and $f_- = 0$ at $p = \infty$. The integrals are taken along circles positively oriented in D, $C_\pm = \{p \in \mathbf{C} : |p| = \tilde{r}_\pm\}$ and $r_- < \tilde{r}_- < \tilde{r}_+ < r_+$. Now consider a function L in $\mathfrak{g}$ that depends on new variables $t = (t_2, t_3, \ldots)$, of the form

$$L(p,x,t) = p + \sum_{j \geqslant 1} u_j(x,t) p^{-j}, \tag{1.1}$$

for which $L_+ = p$, $L_- = -\sum_{j\geqslant 1} u_j(x,t)p^{-j}$. The Lax–Sato equations for the dispersionless KP (dKP) hierarchy [1] are the system of differential equations

$$\frac{\partial L}{\partial t_j} = \{P_j, L\}, \tag{1.2}$$

with $j = 2, 3, \ldots$ when the positive functions P_j are given by $P_j(p, x, t) = L^j\ |_+$. Since the early works in these equations [2–8], the study of their solutions and reductions has been pursued along a number of different lines. Among them we can cite the hydrodynamic reductions and hodograph solutions [9–12], besides factorization methods and twistor constructions [1] [13–16] in which we shall be mainly interested. Recently, a version of the inverse scattering method has been applied to the construction of formal solutions for the dKP equation [17, 18]. We pass to formulate the factorization and twistor constructions along with a new formulation in terms of the Hamilton–Jacobi equation, which will prove useful for the resolution of the initial-value problem of the hierarchy.

The compatibility conditions for the system (1.2) can be conveniently written as the zero-curvature equations $\mathrm{d}\omega_\pm = \frac{1}{2}\{\omega_\pm, \omega_\pm\}$ for the $\mathfrak{g}$-valued differential 1-forms $\omega_+ = \sum_{j\geqslant 2} P_j\, \mathrm{d}t_j$ and $\omega_- = \sum_{j\geqslant 2} N_j\, \mathrm{d}t_j$. These are the positive and negative projections of the form $\omega = \sum_{j\geqslant 2} L^j\, \mathrm{d}t_j$, $\omega = \omega_+ - \omega_-$. In particular, the dKP equation $\left(u_t - \frac{3}{2}uu_x\right)_x = \frac{3}{4}u_{yy}$ follows for the function $u = 2u_1$ where u_1 is the coefficient of p^{-1} in the series for L in (1.1) and $t_2 = y$, $t_3 = t$. The Lax–Sato equations can be integrated, at least formally, in the Lie group G generated by the Hamiltonian flows acting on E through the adjoint representation. For a given function $H(p, x) \in \mathfrak{g}$, let e^{sH} be the associated uniparametric subgroup in G. The adjoint action on E, $(p, x) \to (\mathrm{Ad}\, \mathrm{e}^{sH} p, \mathrm{Ad}\, \mathrm{e}^{sH} x)$, is the solution (p^s, x^s) of the Hamilton equations $\mathrm{d}p/\mathrm{d}s = -H_x$, $\mathrm{d}x/\mathrm{d}s = H_p$ issued from (p, x) at $s = 0$. In particular, $\mathrm{Ad}\, \mathrm{e}^{H} = \mathrm{Ad}\, \mathrm{e}^{sH}$ at $s = 1$. Consider now a G-valued function $\psi(t)$ depending on the variables $t = (t_2, t_3, \ldots)$ and let $f(t)$ be given in $\mathfrak{g}$ by the adjoint representation as $f(t) = (\mathrm{Ad}\, \psi(t))\, f_0$ for $f_0 \in \mathfrak{g}$. In that situation, we can define the differential $(\mathrm{d}\psi)\psi^{-1}$ by the equation $\mathrm{d}f(t) = \{(\mathrm{d}\psi)\psi^{-1}, f(t)\}$. As a result, the zero-curvature 1-forms $\omega_\pm$ can be conveniently represented as the differentials $\omega_\pm = (\mathrm{d}\psi_\pm)\psi_\pm^{-1}$, for functions $\psi_\pm$ on the time variables t. Here, the elements $\psi_\pm(t)$ belong to the subgroups $G_\pm$ of G associated with the subalgebras $\mathfrak{g}_\pm$. In that case, $\psi_\pm$ are the solution of the factorization problem [1, 13, 15, 16]

$$\mathrm{e}^{t(p)}\, \mathrm{e}^{sH} = \psi_-^{-1}\psi_+, \tag{1.3}$$

for a fixed Hamiltonian $H(p, x)$ independent of the time variables. The new complex variable s represents a deformation parameter and we denote by $t(p)$ the function $t(p) = \sum_{j\geqslant 2} t_j p^j$. If we now multiply (1.3) by ψ_- and take the differential of this relation at the identity, the new variable s included, we get

$$(\mathrm{d}\psi_-)\psi_-^{-1} + \mathrm{d}t(L) + \mathrm{d}s\, H(L, M) = (\mathrm{d}\psi_+)\psi_+^{-1}, \tag{1.4}$$

with $L = (\mathrm{Ad}\, \psi_-\, \mathrm{e}^{t(p)})p$ and $M = (\mathrm{Ad}\, \psi_-\, \mathrm{e}^{t(p)})x$, which form a new pair of canonical variables [1]. As a function of the time variables, this L proves to be a solution of the dKP hierarchy (1.2) as a consequence of relation (1.4). In addition, we obtain the equations

$$\frac{\partial L}{\partial s} = \{H_-, L\}, \qquad \frac{\partial M}{\partial s} = \{H_-, M\}, \tag{1.5}$$

which define a $w_{1+\infty}$-symmetry [1] of the hierarchy (1.2). Finally, if $H(p, x) \in \mathfrak{g}_-$, the initial condition for L that follows from (1.3) is

$$\lambda = L|_{t=0} = (\mathrm{Ad}\, \mathrm{e}^{-sH})p = p + sH_x - \frac{s^2}{2}\{H, H_x\} + O(s^3), \tag{1.6}$$

while for M we observe that

$$\mu = M|_{t=0} = (\mathrm{Ad}\, \mathrm{e}^{-sH})x = x - sH_p + \frac{s^2}{2}\{H, H_p\} + O(s^3). \tag{1.7}$$

The canonical pair λ, μ can be equivalently described as a solution to Hamilton's equations for the Hamiltonian $-H(\lambda, \mu)$,

$$\frac{\mathrm{d}\lambda}{\mathrm{d}s} = \frac{\partial H}{\partial \mu}, \qquad \frac{\mathrm{d}\mu}{\mathrm{d}s} = -\frac{\partial H}{\partial \lambda}. \tag{1.8}$$

Thus we see that to define initial conditions $\lambda = L|_{t=0}$, for solutions (1.1) of the dKP hierarchy (1.2), it is enough to consider negative elements

$$H(p,x) = \sum_{j\geqslant 1} H_j(x)p^{-j}. \tag{1.9}$$

Note that the collection of coefficients $H_j(x)$ is equivalent to the coefficients $u_j(x)$ in (1.1) at $t = 0$. In what follows, we assume that the flow (1.6) is defined at $s = 1$.

The potential dKP hierarchy appears to be naturally associated with the generating function for the canonical transformation which takes (p, x) into the new variables (L, M). Its description may be attained by means of a dynamical interpretation of (1.2) [1, 10, 13] as a system of flows on the manifold E. The final result of this elaboration is the Hamilton–Jacobi theory for the Hamiltonian $H(p, x)$. To see that, we define $\Omega = \mathrm{d}t(L) + \mathrm{d}s H(L, M)$, with positive part $\Omega_+ = \sum_{j\geqslant 2} P_j \,\mathrm{d}t_j + \mathrm{d}s H_+ = (\mathrm{d}\psi_+)\psi_+^{-1}$ by virtue of (1.4). The zero-curvature equation for Ω_+ guarantees the compatibility of the Hamiltonian flows

$$\mathrm{d}p + \{\Omega_+, p\} = 0, \qquad \mathrm{d}x + \{\Omega_+, x\} = 0, \tag{1.10}$$

whose solution can be depicted in terms of the factorization problem (1.3) by the formulae

$$p(t,s) = \left(\mathrm{Ad}\,\psi_+^{-1}\right)p, \qquad x(t,s) = \left(\mathrm{Ad}\,\psi_+^{-1}\right)x.$$

These imply for $L(p(t, s), x(t, s), t, s)$ and $M(p(t, s), x(t, s), t, s)$ the conditions

$$\frac{\mathrm{d}L}{\mathrm{d}t_j} = 0, \qquad \frac{\mathrm{d}M}{\mathrm{d}t_j} = 0, \qquad j = 2, 3, \ldots,$$

and also the following equations:

$$\frac{\mathrm{d}L}{\mathrm{d}s} = -\{H(L,M), L\}, \qquad \frac{\mathrm{d}M}{\mathrm{d}s} = -\{H(L,M), M\}$$

as a consequence of, for instance for L, $\mathrm{d}L/\mathrm{d}s = \partial L/\partial s + \{L, H_+\}$ along the flows (1.10) and equations (1.5). Now, we shall consider the Poincaré–Cartan 1-form [19] for the system (1.10) which is given by

$$\mathrm{d}S = p\,\mathrm{d}x + \Omega_+ = p\,\mathrm{d}x + \sum_{j\geqslant 2} P_j\,\mathrm{d}t_j + H_+\,\mathrm{d}s.$$

It follows from the invariance of the action that, according to the evolution we obtained for (L, M), in these new canonical variables the Hamiltonians for the flows are zero, except for the s-flow. Writing the differential of the action $\mathrm{d}S$ in both systems of coordinates, we shall have

$$p\,\mathrm{d}x + \sum_{j\geqslant 2} P_j\,\mathrm{d}t_j + H_+\,\mathrm{d}s = -M\,\mathrm{d}L + H(L,M)\,\mathrm{d}s + \mathrm{d}\Phi.$$

The evolution laws for the generating function $\Phi(L, x, t, s)$ readily follow from this relation,

$$\frac{\partial\Phi}{\partial s} = H(L,M)|_-, \qquad \frac{\partial\Phi}{\partial t_j} = P_j, \qquad j = 2, 3, \ldots, \tag{1.11}$$

which hold besides the defining equations of the canonical transformation, $p = \Phi_x$ and $M = \Phi_L$. To account for the structure of the variables (L, M), as they were previously defined, we must have

$$\Phi(L,x) = t(L) + xL + \phi(L,x), \tag{1.12}$$

where

$$\phi(L,x) = \sum_{j\geqslant 1} \phi_j(x)L^{-j},$$

although the dependence on the time variables is not explicitly displayed. We assume that the generating function Φ is analytical in a domain $\tilde{D} = \{L \in \mathbf{C} : \rho_- < |L| < \rho_+\}$ and we denote by $\gamma_\pm$ the closed curves in $\tilde{D}$ corresponding to $C_\pm$ in D under the transformation. According to (1.12), we arrive at the relations

$$p = \Phi_x = L + \phi_x(L, x), \qquad M = \Phi_L = x + t'(L) + \phi_L(L, x) \tag{1.13}$$

that define the canonical transformation $(p, x) \to (L, M)$. Equations (1.11) for the coefficients ϕ_j, as functions of the variables t, are the potential equations we shall write explicitly in the following section.

The factorization problem (1.3) proves to be equivalent to the twistor equations, defining solutions of the hierarchy around $t = 0$. The equivalence is obtained by means of the canonical variables (P, X) that at each s are defined according to the formulae

$$P(p, x) = \mathrm{Ad}\, \mathrm{e}^{sH} p, \qquad X(p, x) = \mathrm{Ad}\, \mathrm{e}^{sH} x. \tag{1.14}$$

The action on p and x of an equivalent version of (1.3), $\psi_- \, \mathrm{e}^{t(p)} \, \mathrm{e}^{sH} = \psi_+$, leads to the pair of relations $P(L, M) = (\mathrm{Ad}\, \psi_+)p$, $X(L, M) = (\mathrm{Ad}\, \psi_+)x$. We deduce from these expressions that their negative parts must vanish in each case: a condition prescribed by the twistor equations,

$$P(L, M)|_- = 0, \qquad X(L, M)|_- = 0. \tag{1.15}$$

This twistor system is capable of generating concrete solutions for certain canonical transformations $(p, x) \to (P, X)$ [1, 15, 16]. In general, however, it becomes too implicit as to be useful in this respect. Certainly, this is the case for the transformations defined by a general Hamiltonian flow with Hamiltonian $H(p, x)$ as in the factorization problem (1.3). In this work, we shall obtain solutions to the initial-value problem for the dKP hierarchy according to the kind of prescribed initial data under consideration. Hamiltonians of the form (1.9) represent the initial conditions for the factorization problem (1.3) because at $t = 0$, the group element ψ_- coincides with the flow generated by $-H$, $\psi_-|_{t=0} = \exp(-sH)$. As a consequence of the dynamical interpretation of the Lax–Sato equations for the dKP hierarchy we have just seen, the solution to this factorization problem can be given in terms of the generating function Φ in (1.12). We shall characterize such a function Φ by the functional Hamilton–Jacobi equation (2.2), the solutions of which can be represented by a Taylor's series whose coefficients are obtained recurrently. Nevertheless, and regarding the dKP hierarchy, writing the initial condition for the Lax function $\lambda = L|_{t=0}$ amounts to solving the canonical equations (1.8) for the Hamiltonian $-H$. As a variant of this description, we consider the initial conditions for the potential dKP hierarchy, given by the generating function $\Phi^0 = \Phi|_{t=0}$, that arise as solutions of the usual Hamilton–Jacobi equation (2.4). Thus, we cannot expect for Φ anything more explicit than the expression for its corresponding initial condition Φ^0. Even for a simple H, as expression (2.6) shows, this Φ^0 may be quite involved. Alternatively, as far as we are only interested in the final solutions L or Φ we can try to forget about H and use instead the initial values λ or Φ^0 respectively. As a first step in that direction, we obtain the characterization of the function Φ given by formula (3.3). According to this formula, the solution Φ coincides, modulo a term independent of L, with the solution of the Hamilton–Jacobi equation for the Hamiltonian H for an arbitrarily given analytical initial condition $I(L)$. In this context, we can think of the coefficients I_m of Taylor's series of $I(L)$ as a set of variables equivalent to the times t_m used to write the dKP hierarchy. In this sense, formulae (3.11) and (3.12) represent the solution to the potential hierarchy, in a parametric form, in terms of the variables $\{I_m\}_{m=1}^{\infty}$. The transition to a description independent of the Hamiltonian H is made in theorem 4.1 that reformulates proposition 3.1 into the language of the generating functions. Namely, the flow associated with H defines a monoparametric canonical transformation that can be equivalently described by a generating function, precisely the generating function Φ^0 for the initial condition of the potential hierarchy. The main result of this work consists in the characterization of the solution to the potential dKP hierarchy given by formulae (4.4), (4.5). Considered in the twistor's construction context, our solution makes redundant one half of the twistor system. This connection with the twistor equations serves to compare our construction with other known methods for the resolution of the dKP hierarchy.

The content and results of this paper are as follows. In section 2, a series for the generating function Φ is given. We include here an explicit form of the equations of the potential dKP hierarchy and construct the first two terms of the series for a concrete Hamiltonian, the case of a rational function with a simple pole. In addition, in section 3, we are able to obtain the solution in a simpler form by means of a coordinate transformation that makes the content of proposition 3.1. The transformation leads to the usual Hamilton–Jacobi equation for the Hamiltonian which is fixed by the initial condition $L|_{t=0}$. Thus, we recover the series of section 2 for the generating function and find a parametric representation, given by equations (3.11) and (3.12), that solves the problem for the integrable hierarchy. Section 4 contains the formulation of the solution to the initial-value problem in theorem 4.1, independent of the factorization problem. Moreover, a brief discussion on the reduction of the solution to a finite number of variables is illustrated with an example, and finally the connection of our construction with the twistor equations is considered.

2. The solution to the factorization problem

In this section, we assume that a Hamiltonian of the type (1.9) is given. In that case, as explained in the previous section, the generating function Φ (1.12) determines a solution for the factorization problem (1.3), the potential dKP hierarchy we shall write below, (see equations (2.3)) and the Lax equations (1.2). Next, we study the representation of the function Φ as a Taylor series in the auxiliary time s near the origin. As we shall see, this is a series whose coefficients can be obtained recurrently by means of the first of the equations in (1.11) once the value of Φ is fixed at $s = 0$, namely

$$\frac{\partial \Phi}{\partial s} = H(L, \Phi_L)|_-, \qquad \Phi|_{s=0} = xL + t(L). \tag{2.1}$$

According to (1.3), this problem singles out a solution Φ for which the initial condition in L is (1.6). Now we observe that, from the definition of the negative part in the variable p, it follows that

$$H(L, \Phi_L)|_- = \frac{1}{2\pi \mathrm{i}} \int_{C_-} \frac{\tilde{H}(\zeta, x)}{\zeta - p} \, \mathrm{d}\zeta,$$

where $\tilde{H}(p, x) = H(L, \Phi_L)$ through equations (1.13) for fixed x and the variable ζ is defined by $\zeta = \lambda + \phi_x(\lambda, x)$. With these conventions we denote by γ_- the image of C_- under the given transformation, to write equation (2.1) for Φ in the following form:

$$\frac{\partial}{\partial s} \Phi(L, x) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_-} H(\lambda, \Phi_\lambda) \frac{\Phi_{x,\lambda}(\lambda, x)}{\Phi_x(\lambda, x) - \Phi_x(L, x)} \, \mathrm{d}\lambda. \tag{2.2}$$

In agreement with this formula, we can determine recurrently the coefficients F_j, $j = 1, 2, \ldots,$ of the series:

$$\Phi = t(L) + xL + \sum_{j \geqslant 1} \frac{s^j}{j!} F_j,$$

due to the analyticity of the integrand of (2.2) in the variable s at $s = 0$. To check the correctness of the resulting solution and before computing any concrete example, we shall write down the equations of the potential dKP hierarchy. Let C denote a circle positively oriented, centered at the origin and contained in the domain D. From expression (1.12) for Φ and equations (1.11), we have

$$\mathrm{d}\Phi = \sum_{j\geqslant 1} L^j\,\mathrm{d}t_j + \mathrm{d}\phi = \sum_{j\geqslant 1} L^j|_+\,\mathrm{d}t_j$$

if we keep s constant and include $t_1 = x$ [1] among the time variables. This implies for the coefficients ϕ_j of ϕ in (1.12),

$$\sum_{j\geqslant 1} \mathrm{d}\phi_j L^{-j} = \sum_{j\geqslant 1} \mathrm{d}t_j L^j|_- = \sum_{j\geqslant 1} \mathrm{d}t_j (L^j|_+ - L^j).$$

By integrating in the variable p on C, after we take the product with the differentials $\mathrm{d}P_k$ of the positive parts $P_k = L^k|_+$, we get the set of relations

$$\sum_{j\geqslant 1} \mathrm{d}\phi_j \int_C L^{-j}\,\mathrm{d}P_k = -\sum_{j\geqslant 1} \mathrm{d}t_j \int_C L^j\,\mathrm{d}P_k, \qquad k = 1, 2, \ldots.$$

Integration by parts, the decomposition into positive and negative components for $L^k = P_k - N_k$ and the collection of equations $\partial\phi/\partial t_k = N_k$ for $k = 1, 2, \ldots$ lead to the set of relations

$$j\frac{\partial\phi_j}{\partial t_k} = k\frac{\partial\phi_k}{\partial t_j}, \qquad j, k = 1, 2, \ldots,$$

which show the existence of a potential, the τ-function [1], for which $k\phi_k = \partial \ln\tau/\partial t_k$, $k = 1, 2, \ldots$. To write the remaining equations of the hierarchy we begin by considering, instead of $\mathrm{d}P_k$ above, the integral with $\mathrm{d}p^k$, the differential of the positive power p^k of p,

$$\sum_{j\geqslant 1} \mathrm{d}\phi_j \int_C L^{-j}\,\mathrm{d}p^k = -\sum_{j\geqslant 1} \mathrm{d}t_j \int_C L^j\,\mathrm{d}p^k, \qquad k = 1, 2, \ldots.$$

We now take into account the first of the equations in (1.13), so that we can use L as the new integration variable. Then, after integration by parts and application of the residue's theorem, we get the set of equations

$$k\frac{\partial\phi_k}{\partial t_m} + \sum_{j=1}^{k-2} j\frac{\partial\phi_j}{\partial t_m}\,\mathrm{res}_{L=0}\frac{(L+\phi_x)^k}{L^{j+1}} = m\,\mathrm{res}_{L=0} L^{m-1}(L+\phi_x)^k, \tag{2.3}$$

for k and m positive integers. At $k = 1$ we recover part of the equations defining the τ-function; for $k = 2$, when we take the derivative with respect to x, we obtain

$$\frac{1}{2m}\frac{\partial^2\phi_1}{\partial t_2\partial t_m} = \frac{1}{m+1}\frac{\partial^2\phi_1}{\partial t_1\partial t_{m+1}} + \sum_{a+b=m}\frac{1}{2ab}\frac{\partial}{\partial t_1}\left(\frac{\partial\phi_1}{\partial t_a}\frac{\partial\phi_1}{\partial t_b}\right), \qquad m = 2, 3, \ldots.$$

In these equations, at $m = 2$, we have the potential dKP equation for ϕ_1, $\frac{3}{4}\phi_{1yy} = \left(\phi_{1t} + \frac{3}{2}\phi_{1x}^2\right)_x$, which results in the dKP equation for the function $u = -2\phi_{1x}$.

In particular, as a simple example that illustrates most aspects of the preceding construction, we consider the Hamiltonian

$$H(p, x) = \frac{h(x)}{(p-a)}.$$

The function $h(x)$ is analytical at $x = 0$ and we assume $|a|$ is small enough, in order for $H(p, x)$ to be negative in p. Equating the corresponding powers of s in the equation that follows from (2.2) for this H, after substitution of the series for Φ we find at first order

$$F_1(L, x, t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_-} \frac{H(\lambda, x + t'(\lambda))}{\lambda - L}\, \mathrm{d}\lambda = -\frac{h(x + t'(a))}{L - a},$$

while at second order, with the abbreviated notation $h = h(x + t'(a))$, we encounter the expression

$$F_2(L, x, t) = -\frac{hh'}{(L-a)^3} - \frac{hh''t''(a)}{(L-a)^2} - \frac{1}{2(L-a)}[(hh''' + h'h'')t''(a)^2 + (hh'' + h'^2)t'''(a)],$$

as a consequence of the residue's theorem. These formulae define a solution for the potential equations (2.3) up to second order in s. Concretely, the solution ϕ_1 for the potential dKP equation, the residue of $\phi = sF_1 + (s^2/2)F_2 + O(s^3)$ at $L = 0$, is found to be

$$\phi_1 = -sh - \frac{s^2}{4}[(hh''' + h'h'')t''(a)^2 + (hh'' + h'^2)t'''(a)] + O(s^3).$$

Taking the derivative with respect to x of $-2\phi_1$ gives the solution for the dKP equation (see *supra*).

Looking now at the initial condition, we observe that at $t = 0$ the function $\Phi^0(\lambda, x, s) = \Phi|_{t=0}$ solves the equation

$$\frac{\partial \Phi^0}{\partial s} = H\big(\lambda, \Phi^0_\lambda\big)\big|_-.$$

We denote by (λ, μ) the canonical variables (L, M) at $t = 0$ as in (1.6) and (1.7). But at $t = 0$, the generating function takes the form $\Phi^0 = \lambda x + \phi^0(\lambda, x, s)$ as is seen from (1.12). Therefore, Φ^0_λ has no strictly positive powers of λ and the second member of the equation for Φ^0 results in $H\big(\lambda, \Phi^0_\lambda\big)\big|_- = -H\big(\lambda, \Phi^0_\lambda\big)$. Consequently, the initial condition Φ^0 for the generating function Φ in (2.1) is then the solution of the Cauchy problem for the Hamilton–Jacobi equation for the Hamiltonian H,

$$\frac{\partial \Phi^0}{\partial s} + H\big(\lambda, \Phi^0_\lambda\big) = 0, \qquad \Phi^0|_{s=0} = \lambda x. \tag{2.4}$$

Observe here and in what follows that the original momentum p is replaced by the coordinate λ having the conjugate momentum $\mu = \Phi^0_\lambda$. Coming back to the foregoing example we denote by $K(x)$ a primitive of $h(x)$, to write the previous Hamiltonian as $H(p, x) = K'(x)/(p - a)$. Then, at $t = 0$, the solution of Hamilton's equations (1.8) is given by

$$K(\mu) - K(x) = s\frac{K'(x)^2}{(p-a)^2}, \qquad \lambda - a = (p - a)\frac{K'(\mu)}{K'(x)}, \tag{2.5}$$

when $\lambda|_{s=0} = p$ and $\mu|_{s=0} = x$ as imposed by (2.4). The initial condition λ is found by eliminating μ between equations (2.5). Regarding Φ^0, and according to the integration procedure exposed in the following section, we get the expression

$$\Phi^0(\lambda, x, s) = \lambda\mu - a(\mu - x) - 2s\frac{K'(\mu)}{(\lambda - a)}. \tag{2.6}$$

Here μ is defined in terms of (λ, x, s) by $K(\mu) = K(x) + s\frac{K'(\mu)^2}{(\lambda-a)^2}$, as follows from (2.5) after elimination of p. We see that even for a very simple Hamiltonian, as the one considered in the example, the initial conditions cannot be written very explicitly.

3. Hamilton–Jacobi integrability

To proceed further, we shall study the problem (2.1) in terms of the solutions of the Hamilton–Jacobi equation for the Hamiltonian H. As we shall see, that correspondence between the problem (2.1) and the usual Hamilton–Jacobi equation is furnished by a transformation of coordinates. In this direction, we begin with the observation that the problem (2.1), with the notation conventions $x = t_1$ and

$$T(L) = \sum_{m\geqslant 1} t_m L^m,$$

can be rewritten in the following form:

$$\frac{\partial \Phi}{\partial s} + H(L, \Phi_L) = H(L, \Phi_L)|_+, \qquad \Phi|_{s=0} = T(L) \tag{3.1}$$

that proves to be useful in the present context.

Proposition 3.1. *Let $W(L, s)$ be the solution of*

$$\frac{\partial W}{\partial s} + H(L, W_L) = 0, \qquad W|_{s=0} = I(L), \tag{3.2}$$

where $I(L)$ is a given analytical function, $I(L) = \sum_{m\geqslant 1} I_m L^m$. Then, the solution Φ of (3.1) can be written as

$$\Phi = W + \beta, \tag{3.3}$$

under the replacement in W of the function $I(L)$ by an appropriate deformation $I(L, \mathbf{t}, s)$. The function β compensates the term independent of L in Laurent's series for W, since such a term must be absent in Φ.

Proof. After we decompose $H(L, \Phi_L)$ as a sum of positive and negative parts for the variable L, we write the positive part, now in the variable p, in the form

$$H(L, \Phi_L)|_+ = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_+} \frac{H(\lambda, \Phi_\lambda)}{\lambda - L}\bigg|_+ \mathrm{d}\lambda = \frac{1}{2\pi \mathrm{i}} \sum_{k\geqslant 0} L^k \bigg|_+ \int_{\gamma_+} \frac{H(\lambda, \Phi_\lambda)}{\lambda^{k+1}} \mathrm{d}\lambda.$$

This is so because the contribution of the negative part in L of $H(L, \Phi_L)$ to $H(L, \Phi_L)|_+$ is zero since L^{-1}, as defined from (1.1), extends analytically at $p = \infty$. Therefore, we get the equation

$$\frac{\partial \Phi}{\partial s} - \sum_{m\geqslant 1} \alpha_m \frac{\partial \Phi}{\partial t_m} + H(L, \Phi_L) = \alpha_0,$$

for the solution Φ to (3.1). This follows as a consequence of (1.11), the relation $p = \partial\Phi/\partial t_1$ of (1.13) and the definition made for the functionals on Φ,

$$\alpha_k = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_+} \frac{H(\lambda, \Phi_\lambda)}{\lambda^{k+1}} \mathrm{d}\lambda, \qquad k = 0, 1, 2 \ldots.$$

The coefficients α_k are viewed as functions of the coordinates $(\mathbf{t}, s)$, with $\mathbf{t} = (t_1, t_2, t_3, \ldots)$. Now, we assign to the coordinates $(\mathbf{t}, s)$ near the origin the new coordinates $(\mathbf{I}, s)$, $(\mathbf{t}, s) \to (\mathbf{I}, s)$, where $\mathbf{I} = (I_1, I_2, I_3, \ldots)$. Such a correspondence is defined through the solutions of

$$\frac{\partial I_j}{\partial s} - \sum_{m\geqslant 1} \alpha_m \frac{\partial I_j}{\partial t_m} = 0, \qquad j = 1, 2 \ldots$$

with $I_j|_{s=0} = t_j$. The quantities I_j are then the invariants of a certain vector field that allows us to define the new function $\Psi(L, \mathbf{I}, s) = \Phi(L, \mathbf{t}, s)$ that satisfies the equation

$$\frac{\partial \Psi}{\partial s} + H(L, \Psi_L) = A$$

with $A(\mathbf{I}, s) = \alpha_0(\mathbf{t}, s)$. Observe that at $s = 0$, the definition made for Ψ and the conditions $I_j|_{s=0} = t_j$ imply the relation $\Psi(L, \mathbf{t}, 0) = \Phi(L, \mathbf{t}, 0) = T(L)$. Then, in the coordinate system $(\mathbf{I}, s)$ where the equation for Ψ holds, we shall have

$$\Psi(L, \mathbf{I}, 0) = I(L).$$

Now, we chose the function β to compensate the trivial term $A(\mathbf{I}, s)$ independent of L:

$$\beta(\mathbf{I}, s) = \int_0^s A(\mathbf{I}, \xi)\, \mathrm{d}\xi,$$

that is, the solution of

$$\frac{\partial \beta}{\partial s} = A, \qquad \beta|_{s=0} = 0.$$

To conclude, we define $W(L, \mathbf{I}, s) = \Psi(L, \mathbf{I}, s) - \beta(\mathbf{I}, s)$ that, by virtue of the conditions on Ψ and β, satisfies (3.2). Since by construction $\Psi(L, \mathbf{I}, s) = \Phi(L, \mathbf{t}, s)$, formula (3.3) is true. □

Thus, once we solve (3.2) and replace the initial condition $I(L)$ by the function $I(L, s)$ we come to the generating function Φ. The precise form of the transformation $I(L, s) = \sum_{m\geqslant 1} I_m(s) L^m$ is determined by the positive part $T(L)$ of the Laurent series (1.12) for $\Phi(L, \mathbf{t}, s) = T(L) + \phi(L, \mathbf{t}, s)$. To make this correspondence more explicit and to get rid of the function β, let us take the derivative of equation (3.3) with respect to L. The positive part of the Laurent series is then

$$T'(L) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_+} \frac{\mathrm{d}\lambda}{\lambda - L} \frac{\mathrm{d}}{\mathrm{d}\lambda} W(\lambda, s). \tag{3.4}$$

One can see that this formula characterizes, for small s, the function $I(L, s) = T(L) + s\theta_1(L) + \cdots$ through the functional dependence of the solution W of (3.2) on the initial condition $I(L)$. The result follows from the power series expansion for the solution $W(L, s)$ of (3.2). Analogous considerations for the negative part in L of formula (3.3) imply the representation

$$\phi(L, \mathbf{t}, s) = -\frac{1}{2\pi \mathrm{i}} \int_{\gamma_-} \frac{\mathrm{d}\lambda}{\lambda - L} W(\lambda, s). \tag{3.5}$$

The functional dependence of the generating function Φ on the time variables can be further explained through the integration of the Hamilton–Jacobi equation (3.2). As far as we treat the dynamical problem for the Hamiltonian $H(p, x)$ in (1.9), the sought solution corresponds to the canonical variables $(\lambda, \mu) = (L, M)|_{t=0}$. This follows from (1.6) and (1.7) that at $t = 0$, identify ψ_- with the element $\exp(-sH)$. In that case, the solution to the initial-value problem (3.2) can be expressed in the following form [19]:

$$W(\lambda, s) = I(p) + \int_0^s \mathcal{L}. \tag{3.6}$$

In this formula, $\mathcal{L}$ represents the Lagrangian corresponding to the Hamilton function $H(\lambda, \mu)$ through the Legendre transform, $\mathcal{L} = \mu\dot{\lambda} - H$, where $\mu = W_\lambda$ stands for the canonically conjugate momentum of λ. The integral is taken along an integral curve for the motion equations (1.8). This is the integral curve specified by the initial condition in the problem (3.2),

$$\lambda|_{s=0} = p, \qquad \mu|_{s=0} = \left(\frac{\partial W|_{s=0}}{\partial \lambda}\right)_{\lambda=p} = I'(p). \tag{3.7}$$

Because $H(\lambda, \mu)$ does not depend explicitly on s, we shall have

$$W(\lambda, s) = I(p) - sH(p, I'(p)) + \int_p^\lambda \alpha, \tag{3.8}$$

where the 1-form $\alpha = \mu\,\mathrm{d}\lambda$ must be restricted to the curve $H(\lambda, \mu) = H(p, I'(p))$. The relationship between λ and p is fixed by the solution of the motion equations (1.8) with initial conditions (3.7) that imply an expression for λ of the form

$$\lambda = f(p, I'(p), s). \tag{3.9}$$

As a function of the initial point p, the solution $W(\lambda, s)$ given by formula (3.8) can be written in the form

$$W(\lambda, s) = V(p, I(p), I'(p), s). \tag{3.10}$$

This is the sought functional dependence of $W(\lambda, s)$ on $I(p)$, obtained by substitution of λ as given by (3.9) in formula (3.8).

Now, we take into account the explicit representation of λ and $W(\lambda, s)$, as functions of p, in formulae (3.9) and (3.10) respectively. Their substitution in (3.4) and (3.5) gives the solution ϕ in a parametric form. This means that instead of the usual expression of $\phi(L, \mathbf{t}, s)$ as a function of the time variables $\mathbf{t}$, we obtain both ϕ and $\mathbf{t}$ as functions of the variables $\mathbf{I}$. To compute $T'(L)$, we write the integrand in (3.4) in the form $(\lambda - L)^{-1}\,\mathrm{d}W(\lambda, s)$. Now, keeping s fixed, we take the differential of (3.10) and express $\lambda = f(p, I'(p), s)$ in accordance to (3.9). We finally get the formula

$$T'(L) = \frac{1}{2\pi\mathrm{i}} \int_{C_+} \frac{\mathrm{d}p}{f - L} \frac{\mathrm{d}}{\mathrm{d}p} V, \tag{3.11}$$

where we have omitted the arguments of f and V. Analogous reasoning with the formula for the derivative of ϕ with respect to L that follows from (3.3) leads to the expression

$$\frac{\partial}{\partial L}\phi(L, \mathbf{t}, s) = -\frac{1}{2\pi\mathrm{i}} \int_{C_-} \frac{\mathrm{d}p}{f - L} \frac{\mathrm{d}}{\mathrm{d}p} V. \tag{3.12}$$

The relevance of these formulae manifests in the resolution of the initial-value problem for the dKP and potential dKP hierarchies without taking explicitly into account the Hamiltonian H. This will be the subject of the following section where concrete examples will be considered.

4. The initial-value problem

In the previous section, we have seen a method to obtain the transformation defined by a generating function Φ as in (1.12) and solving the potential dKP hierarchy. In this construction, the solution Φ is determined by the Hamiltonian $H(p, x)$ which fixes the factorization problem (1.3). Thus, the preceding construction might be interpreted as the solution of such a factorization problem: to know the transformation ψ_- in (1.3) proves to be equivalent to knowing the generating function Φ in (1.12). From the point of view of the dKP hierarchy, however, instead the Hamiltonian $H(p, x)$ more often than not the solution is fixed by the Lax function $L(p, x, t)$ of (1.1) at $t = 0$. Equivalently, as initial data we might define the expression for the generating function at $t = 0$, $\Phi^0 = \Phi|_{t=0}$. In each case, we have an initial-value problem: for the dKP hierarchy in terms of $L|_{t=0}$ and for the potential dKP hierarchy (2.5) with $\Phi|_{t=0}$. Equivalence among these formulations follows from the constructions given in section 1.

Assume that the Hamiltonian $H(p,x)$ is given and let (λ,μ) denote the canonical pair (L,M), at $t=0$, represented by the part of equations (1.6) and (1.7) given by

$$\lambda = (\mathrm{Ad}\,\mathrm{e}^{-sH})p, \qquad \mu = (\mathrm{Ad}\,\mathrm{e}^{-sH})x. \tag{4.1}$$

Equivalently, we can describe this canonical transformation $(p,x)\to(\lambda,\mu)$ by means of our generating function Φ in (1.12) at $t=0$, $\Phi^0(\lambda,x) = x\lambda + \phi^0(\lambda,x)$. It follows from (1.13) that the equations

$$p = \Phi^0_x = \lambda + \phi^0_x(\lambda,x), \qquad \mu = \Phi^0_\lambda = x + \phi^0_\lambda(\lambda,x) \tag{4.2}$$

determine the new canonical pair from the generating function with the dependence in s implicit in ϕ^0. Now, let us observe that in the construction of section 3, where L is taken as an independent variable, formulae (3.11) and (3.12) define the conjugate momentum $M=\Phi_\lambda$ in terms of initial data already known.

Theorem 4.1. *The generating function $\Phi(L,\mathbf{t})$ can be reconstructed from its initial value $\Phi^0(L,x)$ from the equations*

$$p = \lambda + \phi^0_x(\lambda, I'(p)), \qquad \Phi_\lambda = I'(p) + \phi^0_\lambda(\lambda, I'(p)). \tag{4.3}$$

The parametric representation for the solution Φ is furnished by the condition

$$T'(L) = \frac{1}{2\pi\mathrm{i}}\int_{C_+}\frac{\mathrm{d}p}{\lambda(p)-L}\frac{\mathrm{d}\lambda}{\mathrm{d}p}\big[I'(p)+\phi^0_\lambda(\lambda(p),I'(p))\big], \tag{4.4}$$

and the relation

$$\frac{\partial}{\partial L}\phi(L,\mathbf{t}) = -\frac{1}{2\pi\mathrm{i}}\int_{C_-}\frac{\mathrm{d}p}{\lambda(p)-L}\frac{\mathrm{d}\lambda}{\mathrm{d}p}\big[I'(p)+\phi^0_\lambda(\lambda(p),I'(p))\big], \tag{4.5}$$

where $\lambda(p)$ is defined by the first of equations (4.3) and the function $I'(p)$ is as in section 3.

Proof. The relation $\Phi = W+\beta$, which makes the content of proposition 3.1, implies that under suitable deformations the momentum does not change, $M=\mu$. This happens for initial conditions, $s=0$, on the curve $x=I'(p)$ as in equations (3.7), since by (4.1) $x=\mu|_{s=0}$. Equations (4.3) are the transcription of this property into the language of the transformation (4.2). Positive and negative parts for the Laurent series of Φ_λ become formulae (4.4) and(4.5) by arguments similar to those given in section 3. □

If, instead $\Phi^0(\lambda,x)$, we are given the initial condition $\lambda(p,x)$ for the Lax function $L(p,x,t)$ in (1.1),

$$\lambda = p + \sum_{j\geqslant 1}\frac{u^0_j(x)}{p^j}, \tag{4.6}$$

we can reconstruct the solution $L(p,x,t)$ from the solution Φ determined by the theorem, an initial condition for the generating function is

$$\Phi^0(\lambda,x) = \int p(\lambda,x)\,\mathrm{d}x.$$

In the integrand, the momentum $p(\lambda,x)$ is written as a function of λ and x after inversion of the series (4.6). Let us consider, as an application of the theorem, the problem for the initial condition,

$$\Phi^0(\lambda,x) = \lambda x + s\frac{g(x)}{\lambda-a},$$

where $g(x)$ is analytical at $x = 0$ and the simple pole a is near the origin. Also observe that the deformation parameter s does not coincide with the additional time s used before. The invariant function $I'(p)$, as in section 3, admits a series expansion $I'(p) = T'(p) + s\eta(p) + O(s^2)$. Let $R(L, p)$ denote the integrand in formulae (4.4), (4.5) for the present case:

$$R(L, p) = \frac{\mathrm{d}\lambda/\mathrm{d}p}{\lambda(p) - L}\left[I'(p) - s\frac{g[I'(p)]}{(\lambda(p) - a)^2}\right],$$

with the function $\lambda(p)$ defined by the first of the equations in (4.3),

$$p = \lambda + s\frac{g'[I'(p)]}{\lambda - a}.$$

Since the function $R(L, p)$ has singular points at $p = a$ and $p = L$ inside the disk bounded by C_+, because L may be arbitrarily close to p, by the residue's theorem the integral in formula (4.4) gives

$$T'(L) = \mathrm{res}_{p=a} R(L, p) + \mathrm{res}_{p=L} R(L, p).$$

The Taylor series in s of this expression determines the coefficients of the series for $I'(p)$. In particular, for the first-order term we obtain

$$\eta(p) = -\frac{g'[T'(p)]T''(p)}{p - a} + \frac{g[T'(p)] - g[T'(a)]}{(p - a)^2},$$

which is analytical in p even at $p = a$. Now taking into account the second formula, we shall have

$$\frac{\partial}{\partial L}\phi(L, \mathbf{t}) = -\mathrm{res}_{p=a} R(L, p)$$

because the only singularity that contributes to the integral is found at $p = a$. The Taylor series in s gives for the coefficient ϕ_1 of L^{-1} in $\phi(L, \mathbf{t})$, the solution of the potential dKP equation, the following expression:

$$\phi_1 = sg[T'(a)] + \frac{s^2}{2}g'[T'(a)]g''[T'(a)]T''(a)^2 + O(s^3).$$

Additional orders in s are computed analogously.

The question about the existence of reductions to a finite number of times $t_1, t_2, \ldots, t_{r+1}$ for the solutions studied in this work can be answered affirmatively for the following types of initial data. Consider, for $r \geqslant 2$, initial values for the Lax function (4.6) of the form

$$\lambda(p, x) = p\left(1 + \sum_{j\geqslant 0} p^{-j}a_j(x/p^r)\right).$$

Alternatively, suppose a generating function (1.12) for which $\Phi^0 = \Phi|_{t=0}$ is defined by

$$\Phi^0(L, x) = Lx\left(1 + \sum_{j\geqslant 0} L^{-j}A_j(x/L^r)\right),$$

or that we have a Hamiltonian H as in (1.9) that is given by

$$H(p, x) = px\sum_{j\geqslant 0} p^{-j}h_j(x/p^r),$$

for analytical coefficients a_j, A_j, h_j at the origin. In all three cases, one can see from formulae (4.4) and (4.5) or (3.11) and (3.12) that the associated solutions admit the reduction of the function $I(p)$ to a polynomial of degree $r + 1$ in p, $I(p) = I_1p + I_2p^2 + \cdots + I_{r+1}p^{r+1}$.

In that case, there are exactly $r+1$ nonzero parameters $I_1, I_2, \ldots, I_{r+1}$ in the associated solutions that correspond to the first $r+1$ times that determine the finite parametrization for the solutions. These belong to the type already studied in [15], for generating functions that depend on x mainly through the quotient x/p^r. We shall not dwell upon the general proof of these statements. It consists in showing that λ is of the order of p, and reciprocally once we substitute $x = I'(p)$ which is a polynomial in p of degree r. Instead, we shall concentrate in the following example: let us consider for $r = 2$ the Hamiltonian, $H(p, x) = x^2/p = px(x/p^2)$, which should determine solutions that reduce to the three variables t_1, t_2, t_3. To see that, we begin with the solution to the motion equations (1.8) for $-H$, which leads to the following expression (3.9) for the initial value of the Lax function λ:

$$\lambda = f(p, I'(p), s) = p\left(1 + 3s\frac{I'(p)}{p^2}\right)^{2/3}.$$

A direct application of equation (3.8) yields for the action the following formula:

$$W(L, s) = V(p, I(p), I'(p), s) = I(p) + s\frac{I'(p)^2}{p},$$

with the notation conventions of (3.10). We are now in a position to compute the coefficients t_m by (3.11) according to the expressions

$$mt_m = \operatorname{res}_{p=0} f(p, I'(p), s)^{-m}\frac{\mathrm{d}}{\mathrm{d}p}V(p, I(p), I'(p), s),$$

and analogously for the functions ϕ_m, we get from (3.12)

$$-m\phi_m = \operatorname{res}_{p=0} f(p, I'(p), s)^{m}\frac{\mathrm{d}}{\mathrm{d}p}V(p, I(p), I'(p), s),$$

for $m = 1, 2, \ldots$ in both cases. Until now, these formulae are valid in the general position. At this point, we reduce $I(p)$ to the polynomial $I(p) = I_1 p + I_2 p^2 + I_3 p^3$. In that case, our formulae for t_m give

$$t_3 = \frac{I_3}{1 + 9sI_3}, \qquad t_2 = \frac{I_2}{(1 + 9sI_3)^{2/3}}, \qquad t_1 = \frac{I_1(1 + 9sI_3) - 4sI_2^2}{(1 + 9sI_3)^{5/3}}$$

with the remaining $t_m = 0$ for $m \geqslant 4$, consistent with the previous equations. The solution ϕ_1 to the potential dKP equation in terms of the coordinates I_1, I_2, I_3, in the parametric form, reads

$$\phi_1 = -\frac{16s^3}{3}I_2^4(1 + 9sI_3)^{-\frac{7}{3}} + 4s^2 I_1 I_2^2(1 + 9sI_3)^{-\frac{4}{3}} - sI_1^2(1 + 9sI_3)^{-\frac{1}{3}},$$

while, after the inversion of coordinates, we come to the formula

$$\phi_1 = -\frac{16s^3}{3}t_2^4(1 - 9st_3)^{-3} - 4s^2 t_1 t_2^2(1 - 9st_3)^{-2} - st_1^2(1 - 9st_3)^{-1},$$

valid in our construction for $9|st_3| < 1$.

The system (4.3) may also be studied in the context of the twistor equations (1.15). We observe that the canonical pair (P, X) in (1.14) is defined by the inverse of the transformation (4.1), which fixes the initial conditions $(\lambda, \mu) = (L, M)|_{t=0}$. One can see that in terms of the canonical transformation (4.2), these variables (P, X) are characterized by the equations

$$P = p + \phi_X^0(p, X), \qquad x = X + \phi_p^0(p, X). \tag{4.7}$$

Twistor equations (1.15) are defined by means of $P(L, M)$ and $X(L, M)$, which in the present case are given by the following formulae:

$$P = L + \phi_X^0(L, X), \qquad M = X + \phi_L^0(L, X). \tag{4.8}$$

Because $X_- = 0$ in (1.15), we will have $X = X_+$ and we obtain consequently

$$M = X_+ + \phi_L^0(L, X_+).$$

Thus we see that this equation is nothing else but the second of the equations (4.3) at $L = \lambda$ when we identify $X_+ = I'(p)$. In contrast to the twistor system, in our construction this equation is enough to determine the solution M whereas in the twistor method both L and M must be simultaneously obtained.

References

[1] Takasaki K and Takebe T 1992 *Int. J. Mod. Phys.* A **7** (Suppl. 1B) 889
Takasaki K and Takebe T 1995 *Rev. Math. Phys.* **7** 743
[2] Lebedev D and Manin Yu 1979 *Phys. Lett.* A **74** 154
[3] Zakharov V E 1980 *Funk. Anal. Prilozh.* **14** 89
Zakharov V E 1981 *Physica* D **3** 193
[4] Zakharov V E 1994 Dispersionless limit of integrable systems in 2 + 1 dimensions *Singular Limits of Dispersive Waves (NATO Adv. Sci. Inst. Ser. B Phys.* vol 320*)* ed N M Ercolani *et al* (New York: Plenum)
[5] Krichever I M 1992 *Commun. Pure. Appl. Math.* **47** 437
[6] Krichever I M 1989 *Funct. Anal. Appl.* **22** 200
Krichever I M 1989 *Commun. Math. Phys.* **143** 415
[7] Kupershmidt B A 1990 *J. Phys. A: Math. Gen.* **23** 871
[8] Saveliev M V 1992 *Theor. Math. Phys.* **92** 457
[9] Kodama Y 1988 *Phys. Lett.* A **129** 223
Kodama Y 1988 *Prog. Theor. Phys. Suppl.* **95** 184
[10] Kodama Y and Gibbons J 1989 *Phys. Lett.* A **135** 167
[11] Ferapontov E V and Khusnutdinova K R 2004 *Commun. Math. Phys.* **248** 187
[12] Mañas M, Martinez Alonso L and Medina E 2002 *J. Phys. A: Math. Gen.* **35** 401
[13] Guha P and Takasaki K 1998 *J. Geom. Phys.* **25** 326
[14] Dunajski M, Mason L J and Todd P 2001 *J. Geom. Phys.* **37** 63
[15] Guil F, Mañas M and Martinez Alonso L 2003 *J. Phys. A: Math. Gen.* **36** 6457
[16] Guil F 2004 *J. Phys. A: Math. Gen.* **37** 1
[17] Manakov S V and Santini P M 2006 *Pisma v ZhETF* **83** 534
[18] Manakov S V and Santini P M 2006 *Preprint* nlin.SI/0611047
[19] Arnol'd V I 1989 *Mathematical Methods of Classical Mechanics* 2nd edn (New York: Springer)